

2 Optimality and duality

 Name

Learning objectives

- Derive and apply the optimality conditions for a convex optimization problem
- Formulate the Lagrange dual function and Lagrange dual problem of an optimization problem
- Explain the characteristics and usage of the dual problem
- Describe the weak and strong duality theorems
- Use the optimal values of dual variables for sensitivity analysis
- Formulate the KKT conditions of an optimization problem
- Derive the primal-dual formulation of a convex optimization problem

Bibliography

- Basic reading:
 - Boyd, Stephen P., and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004 (chapters 5).
- Further reading:
 - Motto, A.L.; Arroyo, J.M.; Galiana, F.D., “A Mixed-Integer LP Procedure for the Analysis of Electric Grid Security Under Disruptive Threat,” Power Systems, IEEE Transactions on , vol.20, no.3, pp.1357,1365, Aug. 2005.

Optimality conditions

Write the first-order condition of convexity for a differentiable function f_0 . Then try to infer the criteria for x to be the optimal point of the optimization problem $\min_{x \in X} f_0(x)$, where X is a convex set and $f_0(x)$ a convex function.

Example 2-6: Optimality conditions for generation scheduling

We consider a generating unit with whose production cost can be expressed as $\mathcal{C}(x) = ax^2 + bx + c$, where x represents the production level at MWh, and a , b and c are known parameters. We disregard any limit on the production and denote the electricity price by p . Formulate an optimization problem that determines the optimal generation level to maximize the profit made by the generating unit. Express the optimality condition for this unconstrained optimization problem and discuss its implications.

Solving Example 2-6

Following the optimality condition obtained in the previous exercise, fill in the following table with the optimal production levels for different prices for $a = 0.2$, $b = 10$ and $c = 100$. Afterwards, check your results by solving the code Example1-2.gms for very large values of $\underline{x}$ and $\bar{x}$.

p (€/MWh)	10	20	30	40	50
x (MWh)					

The Lagrangian

Lagrangian of Example 1-2

Assuming $a = 0$, determine the Lagrangian corresponding to Example 1-2. Use $\underline{\eta}$ and $\bar{\eta}$ as the Lagrange multipliers corresponding to production constraints.

Lagrangian of Example 1-3

Assuming $a_g = 0$, determine the Lagrangian corresponding to Example 1-3. Use $\underline{\eta}_g$ and $\bar{\eta}_g$ as the Lagrange multipliers corresponding to production constraints, and λ as the Lagrange multiplier for the balance equation.

The Lagrange dual function

Lagrange dual function of Example 1-2

Lagrange dual function of Example 1-3

Lower bounds on optimal value using the dual function

Lower bound on optimal value of Example 1-2

Solve Example1-2.gms for $a = 0$ and $p = 30$. Chose any “interesting” value of the Lagrange multipliers of Example 1-2. Confirm that the value of the dual function yields a lower bound on the optimal value of the optimization problem.

Lower bound on optimal value of Example 1-3

Solve Example1-3.gms for $a_g = 0$ and $d = 300$. Chose any “interesting” value of the Lagrange multipliers of Example 1-3. Confirm that the value of the dual function yields a lower bound on the optimal value of the optimization problem.

The Lagrange dual problem

Lagrange dual problem of Example 1-2

Lagrange dual problem of Example 1-3

📌 Example 2-7: Solve the dual problem of Example 1-2 (Example2-7.gms)

For $a = 0$ and $p = 30$ solve the dual problem of Example 1-2. Write the optimal values of the dual variables as well as the optimal value.

📌 Example 2-8: Solve the dual problem of Example 1-3 (Example2-8.gms)

For $a_g = 0$ and $d = 300$ solve the dual problem of Example 1-3. Write the optimal values of the dual variables as well as the optimal value.

Weak duality

Strong duality

📌 Example 2-9: Duality as a certificate of optimality of Example 1-2

Solve Example1-2.gms with $a = 0$ and $p = 40$ and write down the optimal values of the dual variables. Check that the dual variables are feasible points of the corresponding dual problem. Check whether the strong duality condition holds in both cases. What does this mean?

📌 Example 2-10: Duality as a certificate of optimality of Example 1-3

Solve Example1-3.gms with $a_g = 0$ and $d = 400$ and write down the optimal values of the dual variables. Check that the dual variables are feasible points of the corresponding dual problem. Check whether the strong duality condition holds in both cases. What does this mean?



The perturbed problem



A global inequality



Local sensitivity analysis and shadow price

📌 Example 2-11: Sensitivity analysis

Using sensitivity theory, determine a lower bound of the objective functions of Example 1-4 if the maximum values of production levels and power flows ($\bar{x}_g$ and $\bar{f}_{nm}$) are increased by 1 MW. Contrast your results by solving the perturbed version of Example4.gms with GAMS.

📌 Example 2-12: Shadow prices

According to the results of Example1-4.gms provided by GAMS, in which technical limit would you prefer to increase to reduce the operation cost of the system?



Complementary slackness



KKT optimality conditions



KKT conditions for convex problems

Example 2-13: KKT conditions of the scheduling problem

Determine the KKT conditions corresponding to Example 1-2. Check that the primal and dual solutions provided by GAMS satisfy these conditions. Use $p = 30$ and the original data corresponding to the generating unit.

Example 2-14: KKT conditions of the economic dispatch

Determine the KKT conditions corresponding to Example 1-3. Check that the primal and dual solutions provided by GAMS satisfy these conditions. Use $d = 300$ and the original data corresponding to the generating units.

Primal-dual formulation

For convex problems, KKT conditions are necessary and sufficient for optimality, which guarantees that strong duality holds. Prove next that imposing the strong duality condition and the rest of KKT conditions, necessarily implies that the complementarity conditions are satisfied. Discuss the advantages of this primal-dual formulation that also provides the optimal solution.

📌 Example 2-15: Primal-dual formulation of the scheduling problem

Determine the primal-dual formulation of the optimization problem in Example 1-2 for $a = 0$. Compare it with its corresponding KKT conditions for $a = 0$.

📌 Example 2-16: Primal-dual formulation of the economic dispatch problem

Determine the primal-dual formulation of the optimization problem in Example 1-3 for $a_g = 0$. Compare it with its corresponding KKT conditions for $a_g = 0$.